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Integral Equation Formulations for Geodetic Mixed Boundary Value Problems

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Abstract

We consider mixed boundary value problems in Physical Geodesy and study possibilities in order to transform them into a system of integral equations over the boundary of the domain. The system of integral equations can be solved numerically, by, e.g. boundary element methods, provided that (a) the mixed boundary value problem is uniquely solvable, (b) the system of integral equations is equivalent to the mixed boundary value problem, and (c) the matrix of integral operators is strongly elliptic. We introduce a method, first proposed by Stephan, which allows to derive integral equation formulations for all mixed boundary value problems relevant to geodetic applications. Moreover, the analysis of Stephan for the mixed Dirichlet-Neumann problem may be generalized to the geodetic mixed boundary value problems, as well.

1 Introduction

The objective of the paper is to study mixed boundary value problems (MBVPs) of type

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D^c \\ B_o u &= g_o \quad \text{on } S_o \\ B_c u &= g_c \quad \text{on } S_c \\ u &= O(|x|^{-1}), \quad |x| \to \infty. \end{aligned} \tag{1}$$

D is a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $S = S_o \cup S_c$, with $S_o \cap S_c = \emptyset$, and D^c its complement in $\mathbb{R}^3$, i.e, $D^c = \mathbb{R}^3 \setminus \bar{D}$. The curve $\bar{S}_o \cap \bar{S}_c$ is assumed to be smooth and simply closed. B_o and B_c are first-order differential operators, and g_o and g_c are the given boundary data. In geodetic applications, S_o can be identified with the surface of the oceans and S_c with the continents. Depending of the choice of B_o and B_c different mixed boundary value problems can be formulated. In geodesy, the most relevant (linearized) MBVPs are summarized in Table 1. Depending on the level of approximation, additional MBVPs can be derived from

Table 1. Linearized geodetic mixed boundary value problems

name	B_o	B_c	type
altimetry-gravimetry I	I	$I - D_\tau$	Dirichlet-Poincaré
altimetry-gravimetry II	D_τ	$I - D_\tau$	oblique-Poincaré
fixed altimetry-gravimetry	I	D_τ	Dirichlet-oblique

the three basic problems listed in Table 1. For instance, in spherical approximation and constant radius approximation, the oblique boundary operator D_τ becomes the Neumann operator, and the Poincaré boundary operator $I - D_\tau$ becomes the Robin operator.

Existence and uniqueness of various linearized geodetic MBVPs have been studied, mostly in the context of the spherical and constant radius approximation, see, e.g., Arnold (1981); Sjöberg (1982); Holota (1982); Svensson (1983); Sacerdote and Sansò (1983a, 1983b); Holota (1983a, 1983b); Arnold (1984); Sacerdote and Sansò (1987); Svensson (1988); Sansò (1993); Keller (1996).

Numerical aspects of geodetic MBVPs have been studied by, e.g., Sjöberg (1982); Bjerhammar (1983); Sansò and Stock (1985); Hofmann-Wellenhof (1985); Mainville (1986); Mayer (1997). In the context of integral equation formulations, the references Sansò and Stock (1985) and Mayer (1997) are of interest. In Sansò and Stock (1985) an integral equation formulation of the linearized altimetry-gravimetry II MBVP in spherical approximation has been used and applied to a local area (see Section 4). The transformation of the MBVP into an integral equation is based on the explicit solution of the Neumann problem for a spherical boundary surface S, and cannot be applied to MBVPs with non-spherical surfaces and/or other types of boundary data. Mayer (1997) proposes a completely new solution strategy for the nonlinear altimetry-gravimetry II MBVP, which assumes a global coverage with gravity values and, in addition, the potential to be given on the free continental part of the boundary. Firstly, a hypersingular integral equation for the linearized fixed gravimetric BVP is solved, due to the global coverage with gravity values. Then, the remaining Dirichlet boundary condition over the free continental part yields a nonlinear operator equation, which has to be solved for the unknown continental geometry. The solution has to be improved iteratively (see Section 5).

Stephan (1987) studied the Dirichlet-Neumann MBVP on closed surfaces in $\mathbb{R}^3$ based on an equivalent formulation of the MBVP as a system of two integral equations. His method is general enough to derive integral equations for all relevant MBVPs in geodesy. Moreover, his procedure to prove the existence and uniqueness of the system of integral equations, and the equivalence of the MBVP with the system of integral equations, may be applied to geodetic MBVP, as well. Therefore we first want to introduce his method and the main lines of the analysis; then we want to show how integral equations for geodetic MBVPs can be derived analogously. Finally, we will briefly discuss the methods of Sansò and Stock (1985) and Mayer (1997) since they also make use of integral equations in order to solve geodetic MBVPs.

2 The method of Stephan

Stephan (1987) discusses the solution of the Dirichlet-Neumann problem in $\mathbb{R}^3$:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D^c \\ u &= g_1 \quad \text{on } S_1 \\ D_n u &= g_2 \quad \text{on } S_2 \\ u &= O(|x|^{-1}), \quad |x| \to \infty. \end{aligned} \tag{2}$$

n is the unit normal vector to S pointing into D^c. S is assumed to be sufficiently smooth. An extension to polyhedral domains is presented in von Petersdorff and Stephan (1990).

Existence and uniqueness of the weak solution u of the Dirichlet-Neumann MBVP (2) is proved by use of (a) the uniqueness of the weak solution, (b) the equivalence of the MBVP to the system of integral equations, (c) the existence and uniqueness of the solution of the system of integral equations, and (d) the solution of the integral equation by inserting into the representation formula. The *weak solution* is defined by Green's first identity: Let $u \in H^1_{loc}(D^c)$ and $v \in H^1(D^c)$ with bounded support. Then

$$\int_{D^c} \nabla u \, \nabla v \, dD = -\int_S \gamma(D_n \, u) \, \gamma v \, dS, \tag{3}$$

where γ denotes the restriction to S. This holds if the trace $\gamma D_n u$ is at least in $H^{-1/2}(S)$. The space U we look for the weak solution is defined by $U := \{u \in H^1_{loc}(D^c) : \Delta u = 0 \text{ in } D^c,\ u = O(|x|^{-1}),\ |x| \to \infty\}$.

The *uniqueness of the weak solution* can easily be proved by means of Green's first identity applied to $\Omega := B \cap D^c$, where B denotes a sufficiently large ball with radius R that encloses $\bar{D}$. Let $u \in U$ with $\gamma_1 u = 0$ and $\gamma_2 D_n u = 0$, where γ_i is the restriction to S_i, $i = \{1, 2\}$. Then

$$\int_{\partial B} u \, D_n u \, d\partial B = -\int_\Omega |\nabla u|^2 \, d\Omega. \tag{4}$$

The left-hand side of (4) tends to zero as $R \to \infty$. This implies $|\nabla u| = 0$, thus $u = const.$ in Ω. Since $u = O(|x|^{-1})$, it follows that $u = 0$.

In order to transform the MBVP (2) into an integral equation we need a representation of the weak solution u of the MBVP in terms of boundary potentials. This can be done in different ways, e.g., by using a representation of u as single layer potential, double layer potential or a linear combination of both. Here, we make use of another possibility, namely of Green's third identity: For $u \in U$, and the *Cauchy-data*

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} := \begin{pmatrix} \gamma u \\ \gamma D_n u \end{pmatrix} \in H^{1/2}(S) \times H^{-1/2}(S), \tag{5}$$

it holds

$$u(x) = \int_S \mu(y) \, D_{n(y)} s(x-y) \, dS(y) - \int_S s(x-y) \, \nu(y) \, dS(y), \quad x \in D^c, \tag{6}$$

where $s(x-y)$ is the fundamental solution of the Laplace equation in $\mathbb{R}^3$, i.e., $s(x-y) = (4\pi\,|y-x|)^{-1}$. The *Calderon-projector*

$$P := \left(\frac{1}{2}I + A\right) \quad \text{with} \quad A := \begin{pmatrix} K & -V \\ D & -K' \end{pmatrix} \tag{7}$$

projects $H^{1/2}(S) \times H^{1/2}(S)$ on the Cauchy-data of weak solutions in U, see Stephan (1987). This projector might be understood as generalization of the well-known limit-relations for the single layer potential and the double layer potential (e.g., Miranda (1970)). Using $P\binom{\mu}{\nu}$ for the Cauchy-data of the weak solution, we obtain the following system of integral equations on S:

$$\frac{1}{2}\begin{pmatrix}\mu \\ \nu\end{pmatrix} = \begin{pmatrix} K & -V \\ D & -K' \end{pmatrix}\begin{pmatrix}\mu \\ \nu\end{pmatrix}. \tag{8}$$

The boundary integral operators V, K, K', and D are defined by ($x \in S$):

$$\begin{aligned}
(V\chi)(x) &= \int_S s(x-y)\,\chi(y)\,dS(y), \\
(K\chi)(x) &= \int_S D_{n(y)} s(x-y)\,\chi(y)\,dS(y), \\
(K'\chi)(x) &= \int_S D_{n(x)} s(x-y)\,\chi(y)\,dS(y), \\
(D\chi)(x) &= \int_S D_{n(x)} D_{n(y)} s(x-y)\,\chi(y)\,dS(y).
\end{aligned} \tag{9}$$

The system (8) together with the boundary conditions in (2) provide more equations than unknowns; depending on how they are combined, we can derive a system of first order integral equations, of second order integral equations, or a mixed system of integral equations. For instance, when restricting the first equation in (8) to S_1 and the second equation to S_2, we obtain

$$\begin{aligned}
&\text{on } S_1\text{: } \frac{1}{2}g_1 = K_{11}g_1 + K_{21}\mu - V_{11}\nu - V_{21}g_2, \\
&\text{on } S_2\text{: } \frac{1}{2}g_2 = D_{12}g_1 + D_{22}\mu - K'_{12}\nu - K'_{22}g_2,
\end{aligned} \tag{10}$$

or, in matrix form,

$$\begin{pmatrix} K_{21} & -V_{11} \\ D_{22} & -K'_{12} \end{pmatrix}\begin{pmatrix}\mu \\ \nu\end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_{11} & V_{21} \\ -D_{12} & \frac{1}{2}I + K'_{22} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{11}$$

The subscript ik means integration over S_i and evaluation on S_k, e.g., if

$$(K\chi)(x) = \int_S D_{n(y)} s(x-y)\,\chi(y)\,dS(y), \quad x \in S, \tag{12}$$

the operator K_{ik} is defined by

$$(K_{ik}\chi)(x) = \int_{S_i} D_{n(y)} s(x-y)\,\chi(y)\,dS(y), \quad x \in S_k. \tag{13}$$

Equation (11) defines a system of first order integral equations for the Cauchy-data $\binom{\mu}{\nu}$. Alternatively, we may restrict the first equation in (8) to S_2 and the second one to S_1; then we obtain

$$\begin{aligned} \text{on } S_1\colon \frac{1}{2}\nu &= D_{11}g_1 + D_{21}\mu - K'_{11}\nu - K'_{21}g_2, \\ \text{on } S_2\colon \frac{1}{2}\mu &= K_{12}g_1 + K_{22}\mu - V_{12}\nu - V_{22}g_2, \end{aligned} \tag{14}$$

i.e., a system of second kind integral equations

$$\begin{pmatrix} \frac{1}{2}I - K_{22} & V_{12} \\ -D_{21} & \frac{1}{2}I + K'_{11} \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} K_{12} & -V_{22} \\ D_{11} & -K'_{21} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{15}$$

Analogously, mixed systems can be obtained by taking only one of the two equations in (8) and restricting first to S_1 and then to S_2.

The *solvability* of the systems of integral equations is shown in several steps. We will omit the details and will only point out the main lines, whereby we limit to system (11). For the details, see Stephan (1987).

Firstly, the mapping properties of the involved integral operators are determined. The operators V, K, and K' have kernels of order $O(|y-x|^{-1})$ as $(y-x) \to 0$, hence they are weakly singular integral operators on S. D has a kernel of order $O(|y-x|^{-3})$ as $(y-x) \to 0$, i.e., it is a hypersingular integral operator on S. Moreover, V, K, K', and D are continuous mappings in suitable Sobolev spaces, i.e., they define pseudodifferential operators of integer order on S. V, K, and K' have order -1, and D has order $+1$. Although the mappings

$$\begin{aligned} V_{ik} &: \tilde{H}^s(S_i) \to H^{s+1}(S_k), \\ K_{ik} &: \tilde{H}^s(S_i) \to H^{s+1}(S_k), \\ K'_{ik} &: \tilde{H}^s(S_i) \to H^{s+1}(S_k), \\ D_{ik} &: \tilde{H}^{s+1}(S_i) \to H^s(S_k), \end{aligned} \tag{16}$$

act only on pieces of S, it can be shown, using standard arguments from the theory of pseudodifferential operators, that they are continuous for some real s, depending on the smoothness of S. Here, $u \in \tilde{H}^s(S_i) = \{u \in H^s(S) : \operatorname{supp} u \subset \bar{S}_i\}$.

Secondly, the system (11) is rewritten in order to make use of the mapping properties (16): if $\tilde{g}_1 \in H^{1/2}(S)$ and $\tilde{g}_2 \in H^{-1/2}(S)$ denote *arbitrary* extensions of the boundary data, the unknown Cauchy-data $\binom{\mu}{\nu}$ admit the form

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} + \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{pmatrix}, \tag{17}$$

with $\mu_0 \in \tilde{H}^{1/2}(S_2)$ and $\nu_0 \in \tilde{H}^{-1/2}(S_1)$ and $\gamma_1\mu_0 = 0$ and $\gamma_2\nu_0 = 0$. Then, (11) can be written as

$$\begin{pmatrix} K_{21} & -V_{11} \\ D_{22} & -K'_{12} \end{pmatrix} \begin{pmatrix} \mu_0 \\ \nu_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_{S1} & V_{S1} \\ -D_{S2} & \frac{1}{2}I + K'_{S2} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \end{pmatrix}, \tag{18}$$

where K_{S1} means integration over S and evaluation on S_1 etc.

Thirdly, the mapping properties of the involved matrix operators

$$A := \begin{pmatrix} K_{21} & -V_{11} \\ D_{22} & -K'_{12} \end{pmatrix}, \qquad B := \begin{pmatrix} \frac{1}{2}I - K_{S1} & V_{S1} \\ -D_{S2} & \frac{1}{2}I + K'_{S2} \end{pmatrix} \tag{19}$$

are investigated. From (16) it follows that the matrix operator A is continuous for some real s depending on the smoothness of S as mapping $\tilde{H}^s(S_2) \times \tilde{H}^s(S_1) \to \tilde{H}^s(S_1) \times \tilde{H}^{s-1}(S_2)$. The continuity of the extensions $\tilde{g}_i$, $i = \{1,2\}$ in $H^s(S)$ for $g_i \in H^s(S_i)$ together with the mapping properties (16), provide the continuity of B as mapping $\tilde{H}^s(S) \times \tilde{H}^{s-1}(S) \to \tilde{H}^s(S_1) \times \tilde{H}^{s-1}(S_2)$, for some real s, depending on the smoothness of S.

Fourthly, the *uniqueness* of (18) is shown. We omit the details and refer to Stephan (1987). Moreover, since the matrix operator A is strongly elliptic, i.e., it satisfies some coerciveness inequalities in appropriate Sobolev spaces, it differs by a compact perturbation from a positive definite operator. Hence, A is a Fredholm operator of index zero. For Fredholm operators of index zero it is known that injectivity implies surjectivity, thus A is bijective.

Finally, the *equivalence* of the original MBVP (2) with the system of integral equations (11) is shown, i.e., $\mu_0 = \gamma_2 u - \gamma_2 \tilde{g}_1$, $\nu_0 = \gamma_1 D_n u - \gamma_1 \tilde{g}_2$, and, conversely, u in D^c is given by

$$u(x) = \int_S \tilde{\mu}(y)\, D_{n(y)} s(x-y)\, dS(y) - \int_S s(x-y)\, \tilde{\nu}(y)\, dS(y), \quad x \in D^c, \tag{20}$$

with

$$\tilde{\mu} = \begin{cases} \mu_0 + \tilde{g}_1 & \text{on } S_2 \\ g_1 & \text{on } S_1 \end{cases}, \quad \tilde{\nu} = \begin{cases} \nu_0 + \tilde{g}_2 & \text{on } S_1 \\ g_2 & \text{on } S_2 \end{cases}, \tag{21}$$

and extensions $\tilde{g}_i$, $i = \{1,2\}$ from above.

3 Application of Stephan's method to geodetic MBVP

The method of Stephan may be applied to any geodetic MBVP in order to transform it into a system of integral equations. Then, we have to study the solvability of the system, making use of the procedure as sketched above, and have to investigate the equivalence of the geodetic MBVP with the system of integral equations. For instance, let us consider the Dirichlet-oblique MBVP

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D^c \\ u &= g_o \quad \text{on } S_o \\ D_\tau u &= g_c \quad \text{on } S_c \\ u &= O(|x|^{-1}), \quad |x| \to \infty, \end{aligned} \tag{22}$$

with Dirichlet data on the oceans and oblique-derivative data on the continents. γ_o and γ_c denotes the restriction to S_o and S_c, respectively. τ defines an oblique unit vector field on S, pointing into D. This problem has been studied by Keller (1996). It results after linearization of the non-linear fixed altimetry-gravimetry MBVP, which assumes that the geometry of the Earth's surface is known and that gravity potential and gravity is given in ocean areas and continental areas, respectively. Keller (1996) shows the existence and uniqueness of the solution using the Kelvin transformation and the Lax-Milgram theorem. In order to transform the Dirichlet-oblique MBVP into an integral equation, we first need a representation formula that connects the Cauchy-data $\alpha := \gamma u$ and $\beta := \frac{1}{\langle n,\tau\rangle}\gamma(D_\tau u)$ with the unknown function u. Starting from Green's third identity (6), we obtain

$$u(x) = \int_S \frac{\alpha(y)}{\langle n,\tau\rangle(y)} D_r s(x-y)\, dS(y) - \int_S s(x-y)\,\beta(y)\, dS(y) + \int_S \langle \epsilon, \nabla(\alpha\, s)\rangle(y)\, dS(y), \quad x \in D^c, \quad (23)$$

with the unit vectors $r = 2\langle n,\tau\rangle\, n - \tau$ and $\epsilon = \frac{\tau}{\langle n,\tau\rangle} - n$. Since $\nabla(\alpha\, s) = Grad(\alpha\, s) + n\, D_n(\alpha\, s)$, and observing that $\langle \epsilon, n\rangle = 0$, we obtain $\langle \epsilon, \nabla(\alpha\, s)\rangle = \langle \epsilon, Grad(\alpha\, s)\rangle$. $Grad$ denotes the surface gradient operator. Moreover, since S is a closed surface, it holds

$$\int_S \langle \epsilon, Grad(\alpha\, s)\rangle\, dS = -\int_S \alpha\, s\, Div\, \epsilon\, dS, \quad (24)$$

with the surface divergence operator Div. Therefore, we obtain for (23)

$$u(x) = \int_S \frac{\alpha(y)}{\langle n,\tau\rangle(y)} D_r s(x-y)\, dS(y) - \int_S s(x-y)\,\beta(y)\, dS(y) - \int_S \alpha(y)\, s(x-y)\,(Div\,\epsilon)(y)\, dS(y), \quad x \in D^c. \quad (25)$$

Equation (25) is our new representation formula. It is called "generalized Green-identity" (cf.Klees (1992, 1997)). Defining the oblique-derivative differential operator

$$P_r := \frac{1}{\langle n,\tau\rangle} D_r - \mathrm{Div}\,\epsilon\, I, \quad (26)$$

we obtain the final form of our representation formula:

$$u(x) = \int_S P_{r(y)} s(x-y)\,\alpha(y)\, dS(y) - \int_S s(x-y)\,\beta(y)\, dS(y), \quad x \in D^c. \quad (27)$$

Observing the jump relations for the single layer potential and its gradient, we obtain the boundary integral equation (cf.Klees (1997))

$$\frac{1}{2}\, u(x) = \int_S P_{r(y)} s(x-y)\,\alpha(y)\, dS(y) - \int_S \beta(y)\, s(x-y)\, dS(y), \quad x \in S. \quad (28)$$

Taking the oblique derivative of (27), we obtain for the limit to the boundary

$$\frac{1}{2}\delta_1\,\beta + \frac{1}{2}\,\delta_2\,\alpha = \int_S D_{\tau(x)} P_{r(y)} s(x-y)\,\alpha(y)\, dS(y) - \int_S \beta(y)\, D_{\tau(x)} s(x-y)\, dS(y), \quad x \in S, \quad (29)$$

with

$$\delta_1 := \langle n, \tau \rangle - 1$$

and

$$\delta_2 := -\left(\langle n, \tau \rangle Div\, \epsilon - \frac{D_\tau \langle n, \tau \rangle}{|\langle n, \tau \rangle|^2} \right).$$

Defining the new boundary integral operators

$$\begin{aligned} (X\chi)(x) &:= \int_S P_{\tau(y)} s(x-y)\, \chi(y)\, dS(y), \\ & x \in S, \\ (U\chi)(x) &:= \int_S D_{\tau(x)} P_{\tau(y)} s(x-y)\, \chi(y)\, dS(y), \\ & x \in S, \\ (W\chi)(x) &:= \int_S D_{\tau(x)} s(x-y)\, \chi(y)\, dS(y), \\ & x \in S, \end{aligned} \tag{30}$$

we obtain the following system of integral equations by restricting (28) first to S_o and then to S_c:

$$\begin{pmatrix} X_{co} & -V_{oo} \\ \frac{1}{2}I - X_{cc} & V_{oc} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - X_{oo} & V_{co} \\ X_{oc} & -V_{cc} \end{pmatrix} \begin{pmatrix} g_o \\ \frac{1}{\langle n, \tau \rangle} g_c \end{pmatrix}. \tag{31}$$

Equation (31) defines a mixed system of boundary integral equations; it is of the second kind w.r.t. α and of the first kind w.r.t. β. We can derive alternative integral equations, e.g., by restricting (28) to S_o and (29) to S_c and vice versa. For instance, restricting (28) to S_c and (29) to S_o, we obtain a system of second kind integral equations for the unknowns α and β:

$$\begin{pmatrix} U_{co} & -W_{oo} - \frac{1}{2}\langle n, \tau \rangle \delta_1 I \\ \frac{1}{2}I - X_{cc} & V_{oc} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\delta_2 I - U_{oo} & W_{co} \\ X_{oc} & -V_{cc} \end{pmatrix} \begin{pmatrix} g_o \\ \frac{1}{\langle n, \tau \rangle} g_c \end{pmatrix}, \tag{32}$$

Analogously, restricting (28) to S_o and (29) to S_c, we obtain the mixed system of integral equations:

$$\begin{pmatrix} \frac{1}{2}\delta_2 I - U_{cc} & W_{oc} \\ X_{co} & -V_{oo} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} U_{oc} & -\frac{1}{2}\langle n, \tau \rangle \delta_1 I - W_{cc} \\ \frac{1}{2} I - X_{oo} & V_{co} \end{pmatrix} \begin{pmatrix} g_o \\ \frac{1}{\langle n, \tau \rangle} g_c \end{pmatrix}. \tag{33}$$

Like (31) it is of the second kind w.r.t. α and of the first kind w.r.t. β. The boundary operators in (31),(32), and (33) have the following mapping properties: For some real s, depending on the smoothness of S, and $i, k = \{o, c\}$, the mappings

$$\begin{aligned} X_{ik} &: \tilde{H}^s(S_i) \to H^s(S_k), \\ U_{ik} &: \tilde{H}^{s+1}(S_i) \to H^s(S_k), \\ W_{ik} &: \tilde{H}^s(S_i) \to H^s(S_k) \end{aligned} \tag{34}$$

are continuous. X_{ik} and W_{ik} define strongly singular integral operators, which are pseudodifferential operators of order 0; U_{ik} is a hypersingular integral operator, which has order 1. For the property of V_{ik}, see (9). What still has to be done is to investigate the solvability of the systems (31)-(33) and to prove the equivalence of the Dirichlet-oblique MBVP with the systems of integral equations. This can be done following the procedure of Stephan (1987).

4 The method of Sansò and Stock

Sansò and Stock (1985) consider the Robin-Neumann mixed boundary value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D^c \\ D_n u &= g_o \quad \text{on } S_o \\ D_n u + \frac{2}{R} u &= g_c \quad \text{on } S_c \\ u &= O(|x|^{-1}), \quad |x| \to \infty, \end{aligned} \tag{35}$$

where S is the surface of a sphere with radius R. They look for a solution $u \in H^{\lambda}_{loc}(S)$ for given boundary data $g_o \in H^{\lambda-1}(S_o)$ and $g_c \in H^{\lambda-1}(S_c)$ with $\frac{1}{2} < \lambda < \frac{3}{2}$. The transformation into a boundary integral equation is based on the explicit solution of the Neumann BVP for a sphere, which is known as Hotine's formula:

$$u(x) = -\frac{R}{4\pi} \int_S H(x-y)\, (\gamma D_n u)(y)\, dS(y), \quad x \in S, \tag{36}$$

with the Green function of the second kind (Hotine function, Neumann function)

$$H(x-y) = \frac{2R}{|x-y|} - \ln\left(1 + \frac{2R}{|x-y|}\right). \tag{37}$$

Defining the integral operator

$$E\,\chi(x) := -\frac{R}{4\pi} \int_S H(x-y)\, \chi(y)\, dS(y), \quad x \in S, \tag{38}$$

equation (36) can be written as $\mu = E\nu$. As in Section 2, $\binom{\mu}{\nu}$ define the traces on S $\binom{\gamma u}{\gamma D_n u}$. Observing the boundary condition of the Robin-Neumann MBVP, we have

$$\mu = E\nu = E_{oS}\nu + E_{cS}\nu = E_{oS} g_o + E_{cS}\left(g_c - \frac{2}{R}\mu\right), \tag{39}$$

hence,

$$\left(I + \frac{2}{R} E_{cS}\right) \mu = E_{oS} g_o. \tag{40}$$

Equation (40) is an integral equation of the second kind for the unknown Cauchy-data μ. Substituting $\mu = \mu_0 + \frac{R}{2} l g_o$, we obtain

$$A\mu_0 := \left(I + \frac{2}{R} E_{cS}\right) \mu_0 = -\left(\frac{R}{2} I - E\right) l g_o =: B l g_o. \tag{41}$$

The operators $A, B : H^s(S) \to H^s(S)$ are continuous for some real s depending on the smoothness of S. Analogously to Section 2, we can prove the solvability of (41) and the equivalence of the integral equation with the Robin-Neumann MBVP. We omit the details. In order to solve the integral equation, we can apply, e.g., the Nyström method, collocation boundary element methods or Galerkin boundary element methods.

The method relies on the spherical topology since only then the Neumann function is known, i.e., there is an explicit solution of the Neumann problem available. However, the basic idea can easily be generalized if instead of Hotine's formula Green's third identity is used. With the results of Section 2, we have

$$\frac{1}{2}\mu = K\mu - V\nu = K\mu - V_{oS}\nu - V_{cS}\nu = K\mu - V_{oS}\nu - V_{cS}\left(g_c - \frac{2}{R}\mu\right), \tag{42}$$

hence,

$$\left(\frac{1}{2}I - K - \frac{2}{R}V_{cS}\right)\mu = -V_{oS}g_o - V_{cS}g_c. \tag{43}$$

The operator $A := \left(\frac{1}{2}I - K - \frac{2}{R}V_{cS}\right)$ is continuous from $H^s(S) \to H^s(S)$, the operator $B := -\binom{V_{oS}}{V_{cS}}^T$ is continuous from $H^{s-1}(S_o) \times H^{s-1}(S_c) \to H^s(S)$.

Alternative integral equations can be derived in different ways making use of (8) and restricting to S_o or S_c. If on S_c $\gamma_c\nu$ is replaced by $g_c - \frac{2}{R}\gamma_c\mu$, we obtain only one integral equation for the unknown $\mu = \gamma u$. For instance, when restricting the first equation in (8) to S_o, we obtain the second kind integral equation

$$\left(\frac{1}{2}I - K_{So} - \frac{2}{R}V_{co}\right)\mu = -V_{oo}g_o - V_{co}g_c, \quad x \in S_o, \tag{44}$$

with weakly singular kernels. When restricting the second equation in (8) to S_c, we obtain a second kind integral equation with weakly singular and hypersingular kernels

$$\left(\frac{1}{2}I + \frac{2}{R}K'_{cc} - D_{Sc}\right)\mu = \left(\frac{1}{2}I + K'_{cc}\right)g_c + K'_{oc}g_o, \quad x \in S_c. \tag{45}$$

We can also derive a first order integral equation by restricting the second equation in (8) to S_o:

$$\left(D_{So} - \frac{2}{R}K'_{co}\right)\mu = -\left(\frac{1}{2}I + K'_{oo}\right)g_o + K'_{co}g_c, \quad x \in S_o. \tag{46}$$

The corresponding kernels are weakly singular and hypersingular. What remains is to prove the solvability of the integral equations and the equivalence with the original MBVP. The prove can easily be done using the procedure and results of Section 2. We omit the details.

5 The method of Mayer

Mayer (1997) considers the altimetry-gravimetry II MBVP in non-linear form. There are two sources of nonlinearities:

1. gravity is a non-linear functional of the potential, and
2. the boundary surface is partly free (over the continents, unobservable by altimeter radar).

The new idea of Mayer is to perform a linearization with respect to source 1 (gravity) only, and later on, to solve the resulting (still non-linear) problem by a special iteration procedure. This approach is justified by recent findings of Heck and Seitz (1993), that source 2 (free boundary) is the more severe source of non-linearity in geodetic boundary value problems. Consequently, if any iteration will be necessary, then w.r.t. source 2.

The formulation of the partly linearized altimetry-gravimetry II MBVP is:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D^c \\ D_\tau u &= g \quad \text{on } S \\ u &= g_c \quad \text{on } S_c \\ u &= O(|x|^{-1}), \quad |x| \to \infty. \end{aligned} \tag{47}$$

The oceanic surface $S_o := S \setminus S_c$ is assumed to be known, as well. If in addition S_c were known instead of g_c, the resulting BVP would be identical to the linearized fixed gravimetric BVP, which in turn equals the classical oblique BVP for the Laplace equation. This problem is much easier to solve because it is *not mixed.* Some theoretical results exist (e.g., Klees (1992)), and have been augmented recently by Mayer (1997). Also from the numerical point of view, this problem is solvable using boundary element methods (e.g., Klees (1992)).

A boundary integral equation for the linearized fixed gravimetric BVP is derived from a representation formula, for which Mayer prefers a combined double- and single-layer potential

$$u(x) = (K\chi)(x) + \kappa(V\chi)(x), \quad x \in D^c, \tag{48}$$

where κ is an arbitrary positive real number, and χ is the surface density. Defining the operators

$$\begin{aligned} (Y\chi(x) &:= \int_S D_{\tau(x)} D_{n(y)} s(x-y)\, \chi(y)\, dS(y), \\ & \quad x \in S \\ (Z\chi(x) &:= \frac{1}{2}\langle Grad\chi, \tau\rangle, \end{aligned} \tag{49}$$

we obtain an integro-differential equation of the second kind on S

$$A\chi := \left(-\frac{1}{2}\kappa\langle n, \tau\rangle I + Z + Y + \kappa W\right)\chi = g. \tag{50}$$

Formally, the solution of this integro-differential equation can be written as

$$\chi = A^{-1} g. \tag{51}$$

Hence, the desired potential function is

$$u(x) = ([K + \kappa V]A^{-1}g)(x), \quad x \in D^c. \tag{52}$$

Now, we return to the actual problem (47), where additionally the Dirichlet condition over the continents

$$\gamma_c u = g_c \tag{53}$$

has to be fulfilled. The only unknown of this equation is the boundary surface S_c. Therefore, we end up with an operator equation, which must be solved for S_c:

$$Q(S_c, g) := \gamma_c u = \gamma_c\{(K + \kappa V)A^{-1}g\} = g_c, \quad \text{on } S_c \tag{54}$$

The operator $Q(S_c, g)$ is *non-linear* in the first argument. The complicated structure of this operator is the price we have to pay for the striking simplicity of the first solution step (51),(52). The nonlinear operator equation (54) must be solved iteratively, starting from an initial guess $\tilde{S}_c$ for S_c, which in geodesy is known as the telluroid. However, note that $\tilde{S} := \tilde{S}_c \cup S_o$ must be a *closed* surface, which is not guaranteed by classical definitions of the telluroid. Quite formally, let us express the *Fréchet expansion* of Q as

$$Q(S_c, g) = Q(\tilde{S}_c, g) + \frac{\partial Q}{\partial S_c}(\tilde{S}_c, g)(S_c - \tilde{S}_c) + o(|S_c - \tilde{S}_c|). \tag{55}$$

This expansion suggests an iterative procedure of Newton type: Let $S_c^{(0)} := \tilde{S}_c$; for $n = 1, 2, \ldots$:

$$S_c^{(n)} := S_c^{(n-1)} + \left(\frac{\partial Q}{\partial S_c}\left(S_c^{(n-1)}, g\right)\right)^{-1}\left(g_c - Q(S_c^{(n-1)}, g)\right). \tag{56}$$

So far, nothing can be said about the feasibility of this suggestion, neither about the *existence and uniqueness* of the inverse Fréchet derivative nor about the *convergence* of this procedure. Moreover, due to the complicated structure of Q there is even less hope to obtain similar results as with classical geodetic approaches. Mayer (1997) has also derived an *explicit expression* for the Fréchet derivative of Q. However, the complexity of this expression will certainly prevent any practical application in geodesy.

Nonetheless, Mayer (1997) has shown that there always exist interesting alternatives to the standard geodetic techniques.

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